

Maximal mutual correlation: an algebraic measure of correlations in bipartite quantum systems

Lech Jakóbczyk *

Institute of Theoretical Physics

University of Wrocław

Plac Maxa Born 9, 50-204 Wrocław, Poland

We introduce an algebraic measure of correlations in bipartite quantum systems. The proposed quantity, called maximal mutual correlation, provides the information how much a given state differs from the product state of its marginals. In contrast to the entanglement or quantum discord, maximal mutual correlation can be non - zero for some classically correlated quantum states.

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I. INTRODUCTION

Various notions of quantumness of correlations in a given quantum state give essentially the same information in the case of pure states. A pure state of bipartite quantum system is either separable (i.e. a product state) or nonseparable. This mathematical notion of nonseparability can be extended to the mixed states, giving the natural (but formal) definition of entanglement of general quantum states: a (mixed) state is entangled if it cannot be expressed as a convex combination of pure separable states [1]. It was widely accepted that such defined entanglement is the only source of quantumness of correlations. In the recent years however, other, more general features of correlations have attracted much interest. They arise from the observation that for pure separable state, there exists von Neumann measurement on a part of composite system that do not disturb the state, whereas nonseparable states are always disturbed by such local measurements. Extension of this feature to the mixed states, gives rise of the notion of quantum discord [2–4]. For pure states discord coincides with entanglement, but in the case of mixed states discord and entanglement differ significantly. For example, almost all quantum states have non-vanishing discord and in particular there exist discordant separable mixed states [5].

A pure separable state of bipartite systems has also important algebraic property: the expectation value ω in such state factorizes i.e.

$$\omega(AB) = \omega(A)\omega(B) \quad (I.1)$$

for the product of all local observables. On the other hand, for nonseparable states there are such pairs of local observables that (I.1) is not valid. Therefore the maximum of the numbers

$$|\omega(AB) - \omega(A)\omega(B)| \quad (I.2)$$

taken over all pairs of local observables, can serve as a kind of an algebraic measure of quantumness of correlations. This idea was applied to the case of two qubits in Ref. [6], where it was shown that the suitably normalized maximum of (I.2)

exactly equals to the pure state entanglement. In the present note we study the extension of that measure of correlations to the mixed two-qubit states. The extended measure we call *maximal mutual correlation* between subsystems in a given state. In contrast to the entanglement or quantum discord, this quantity can be non - zero even for classically correlated (i.e. non - discordant) quantum states.

In this article we compute maximal mutual correlation for several classes of states and study the relation of this quantity to other measures of correlations in two - qubit quantum systems. In particular we show that in the case of so called "classical - quantum" states, this measure depends on the classical probability distribution defining such states but also encodes some information about quantum states of the second subsystem, whereas for "classical - classical" states it depends only on the properties of the corresponding classical probability distribution. In the states containing some quantum correlations, maximal mutual correlation is always greater or equal to geometric measure of quantum discord. This relation is proved to be valid in the class of Bell - diagonal states and it is conjectured that is also true for general two - qubit states.

II. BIPARTITE CORRELATIONS IN THE ALGEBRAIC FRAMEWORK

A. General discussion

For the unambiguous discussion of quantum non-separability, the crucial is a clear identification of subsystems of the total system. In the context of algebraic quantum theory such identification is given by the proper choice of the factorization of the total algebra of observables into subalgebras describing subsystems (see e.g. [7]). In the algebraic formulation of quantum theory (see e.g. [8]), the total system is described by the $*$ - algebra \mathcal{A}_{tot} of all observables. Since in the present note we consider only the systems with finite number of levels, \mathcal{A}_{tot} is isomorphic to the full matrix algebra. The most general state on \mathcal{A}_{tot} is given by the linear functional

$$\omega : \mathcal{A}_{\text{tot}} \rightarrow \mathbb{C}, \quad (II.1)$$

which is positive i.e. for all $A \in \mathcal{A}_{\text{tot}}$

$$\omega(A^*A) \geq 0,$$

*ljak@ift.uni.wroc.pl

and normalized i.e.

$$\omega(\mathbb{1}) = 1.$$

For the matrix algebras, any state ω is normal i.e. there exists density matrix ρ such that

$$\omega(A) = \text{tr}(\rho A) \quad (\text{II.2})$$

To describe subsystems of the total system, we consider subalgebras \mathcal{A} and \mathcal{B} of \mathcal{A}_{tot} . We assume that $\mathcal{A} \cap \mathcal{B} = \{c\mathbb{1}\}$ and $\mathcal{A}_{\text{tot}} = \mathcal{A} \vee \mathcal{B}$ i.e. \mathcal{A}_{tot} is generated by \mathcal{A} and \mathcal{B} . For the discussion of correlations between observables in \mathcal{A} and \mathcal{B} in a given state ω , the crucial is the notion of uncorrelated states. The formal definition is as follows: the state ω on $\mathcal{A} \vee \mathcal{B}$ is $(\mathcal{A}, \mathcal{B})$ - *uncorrelated* if

$$\omega(AB) = \omega(A)\omega(B) \quad (\text{II.3})$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. As it was shown in Ref. [9] the very existence of such uncorrelated or product states depends on the mutual commutativity of the algebras \mathcal{A} and \mathcal{B} . Thus in the context of the theory of correlations in bipartite systems, the total algebra of observables should be such that $\mathcal{A}_{\text{tot}} = \mathcal{A} \vee \mathcal{B}$, where $\mathcal{A} \cap \mathcal{B} = \{c\mathbb{1}\}$ and for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $[A, B] = 0$. In the case of full matrix algebras, much more can be said about the structure of \mathcal{A}_{tot} . One can show that $\mathcal{A} \vee \mathcal{B}$ is isomorphic to the tensor product $\mathcal{A} \otimes \mathcal{B}$. So \mathcal{A} is generated by elements $A \otimes \mathbb{1}$ and \mathcal{B} is generated by $\mathbb{1} \otimes B$.

Now we pass to the discussion of correlations between subsystems. A state ω on \mathcal{A}_{tot} is $(\mathcal{A}, \mathcal{B})$ - *correlated* if it is not a product state with respect to subalgebras \mathcal{A} and \mathcal{B} . Next we introduce a quantity which we call maximal mutual correlation between \mathcal{A} and \mathcal{B} in ω . It is defined as [6]

$$C_M(\omega) = \sup_{A, B} |\omega(AB) - \omega(A)\omega(B)|. \quad (\text{II.4})$$

In the formula (II.4), the supremum is taken over all normalized elements $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The quantity $C_M(\omega)$ gives the information how much a state differs from the product state of its marginals, when we take into account only local measurements. Since the states ω are of the form (II.2), we put $C_M(\omega) = C_M(\rho)$. Notice that

$$C_M(\rho) \leq \|\omega - \omega^{\mathcal{A}} \otimes \omega^{\mathcal{B}}\| = \|\rho - \rho^{\mathcal{A}} \otimes \rho^{\mathcal{B}}\|_1, \quad (\text{II.5})$$

where $\|\cdot\|_1$ is a trace norm and the marginal states $\omega^{\mathcal{A}}$ and $\omega^{\mathcal{B}}$ are given by partial traces $\rho^{\mathcal{A}}$ and $\rho^{\mathcal{B}}$ of a density matrix ρ . Thus $C_M(\rho)$ is always dominated by so called *correlation distance* $C(\rho)$, given by

$$C(\rho) = \|\rho - \rho^{\mathcal{A}} \otimes \rho^{\mathcal{B}}\|_1 \quad (\text{II.6})$$

and introduced recently by Hall [10] in the context of the analysis of quantum mutual information.

B. Two qubit case

In the case of two qubits, the total algebra \mathcal{A}_{tot} can be considered as generated by matrix unit $\mathbb{1}$ and elements

$\lambda_1, \dots, \lambda_{15}$, where

$$\lambda_i = \mathbb{1} \otimes \sigma_i, \quad \lambda_{3+i} = \sigma_i \otimes \mathbb{1}, \quad i = 1, 2, 3 \quad (\text{II.7})$$

and λ_j , $j = 7, \dots, 15$ are given by tensor products of the Pauli matrices σ_i taken in the lexicographical order. So

$$\mathcal{A}_{\text{tot}} = [\mathbb{1}, \lambda_1, \dots, \lambda_{15}], \quad (\text{II.8})$$

and

$$\mathcal{A} = [\mathbb{1}, A_1, A_2, A_3], \quad \mathcal{B} = [\mathbb{1}, B_1, B_2, B_3], \quad (\text{II.9})$$

where

$$A_i = \lambda_{3+i}, \quad B_i = \lambda_i, \quad i = 1, 2, 3. \quad (\text{II.10})$$

It is convenient to take elements $A \in \mathcal{A}$ and $B \in \mathcal{B}$ defined as

$$A = a_1 A_1 + a_2 A_2 + a_3 A_3, \quad B = b_1 B_1 + b_2 B_2 + b_3 B_3, \quad (\text{II.11})$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ are normalized vectors in \mathbb{R}^3 . Then $A^2 = \mathbb{1}$, $B^2 = \mathbb{1}$ and

$$\omega(AB) - \omega(A)\omega(B) = \langle \mathbf{a}, Q\mathbf{b} \rangle, \quad (\text{II.12})$$

where the covariance matrix $Q = (q_{ij})$ has the matrix elements

$$q_{ij} = \text{tr}(\rho A_i B_j) - \text{tr}(\rho A_i) \text{tr}(\rho B_j). \quad (\text{II.13})$$

So [6]

$$C_M(\rho) = \sup_{\|\mathbf{a}\|=\|\mathbf{b}\|=1} |\langle \mathbf{a}, Q\mathbf{b} \rangle|. \quad (\text{II.14})$$

Notice that the right hand side of (II.14) is the norm of the matrix Q , thus for two qubits

$$C_M(\rho) = \max(t_1, t_2, t_3), \quad (\text{II.15})$$

where $\{t_1, t_2, t_3\}$ are singular values of the covariance matrix Q .

In the two - qubit case, there is also an explicit expression for the correlation distance $C(\rho)$ [10]. Again it is given in terms of singular values of Q

$$C(\rho) = \frac{1}{4} (|t_1 + t_2 + t_3| + |t_1 + t_2 - t_3| + |t_1 - t_2 + t_3| + |t_1 - t_2 - t_3|) \quad (\text{II.16})$$

III. RELATION OF $C_M(\rho)$ TO OTHER MEASURES OF CORRELATIONS

In the general case, the maximal mutual correlation $C_M(\rho)$ is upper bounded by the correlation distance $C(\rho)$. In this section we focus on the case of two qubits and compare $C_M(\rho)$ with other measures of correlations corresponding to entanglement and quantum discord. As a measure of entanglement we choose normalized negativity $N(\rho)$, given by

$$N(\rho) = \|\rho^{PT}\|_1 - 1, \quad (\text{III.1})$$

where PT stands for partial transposition. To quantify quantum discord, we take geometric measure of quantum discord defined by using trace norm distance in the set of states i.e. the quantity $D_1(\rho)$ defined as [11]

$$D_1(\rho) = \min_{\mathbb{P}^{\mathcal{A}}} \|\rho - \mathbb{P}^{\mathcal{A}}(\rho)\|_1, \quad (\text{III.2})$$

where $\mathbb{P}^{\mathcal{A}}$ is the projective measurement on subsystem \mathcal{A} i.e.

$$\mathbb{P}^{\mathcal{A}}(\rho) = \sum_k (P_k \otimes \mathbb{1}) \rho (P_k \otimes \mathbb{1}). \quad (\text{III.3})$$

Analytic expression for D_1 is known only for Bell - diagonal and X - shaped mixed states [12]. In this note we study X - shaped two - qubit states

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \quad (\text{III.4})$$

where all matrix elements are real and non - negative. The quantity D_1 for such states can be computed as follows. Let $x = 2(\rho_{11} + \rho_{22}) - 1$ and

$$\alpha_1 = 2(\rho_{23} + \rho_{14}), \quad \alpha_2 = 2(\rho_{23} - \rho_{14}), \quad \alpha_3 = 1 - 2(\rho_{22} + \rho_{33}). \quad (\text{III.5})$$

Then [12]

$$D_1(\rho) = \sqrt{\frac{a\alpha_1^2 - b\alpha_2^2}{a - b + \alpha_1^2 - \alpha_2^2}}, \quad (\text{III.6})$$

where

$$a = \max(\alpha_3^2, \alpha_2^2 + x^2), \quad b = \min(\alpha_3^2, \alpha_1^2). \quad (\text{III.7})$$

Notice that the formula (III.6) is not valid in the case when $x = 0$ and

$$|\alpha_1| = |\alpha_2| = |\alpha_3|. \quad (\text{III.8})$$

In such a case, one can use general prescription how to compute D_1 , also given in Ref. [12] (eq. (65)).

Now we go to study the relations between $N(\rho)$, $D_1(\rho)$ and $C_M(\rho)$ for some specific classes of states of two qubits.

A. Pure two qubit states

Let us consider first the simple case of pure two qubit states. Up to the local unitary operations, the corresponding density matrix ρ_{pure} can be written as

$$\rho_{\text{pure}} = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{1 - N^2} & 0 & 0 & N \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ N & 0 & 0 & 1 - \sqrt{1 - N^2} \end{pmatrix}, \quad (\text{III.9})$$

where $N \in [0, 1]$. It is straightforward to show that in this case

$$N(\rho_{\text{pure}}) = D_1(\rho_{\text{pure}}) = N. \quad (\text{III.10})$$

Moreover the corresponding covariance matrix is of the form

$$Q = \begin{pmatrix} N & 0 & 0 \\ 0 & -N & 0 \\ 0 & 0 & N^2 \end{pmatrix}, \quad (\text{III.11})$$

so

$$C_M(\rho_{\text{pure}}) = \|Q\| = N, \quad (\text{III.12})$$

and

$$C(\rho_{\text{pure}}) = N + \frac{1}{2}N^2. \quad (\text{III.13})$$

Thus we arrive at the conclusion that for the correlated two qubit pure states

$$N(\rho_{\text{pure}}) = D_1(\rho_{\text{pure}}) = C_M(\rho_{\text{pure}}) < C(\rho_{\text{pure}}). \quad (\text{III.14})$$

B. Classically correlated mixed two qubit states

We start the analysis of mixed two qubit states by considering so called classical - quantum and classical - classical states. When we consider projective measurements on subsystem \mathcal{A} , there is a subclass of states which are left unperturbed by at least one such measurement. Such "classical - quantum" states ρ_{cq} have zero discord and are of the form

$$\rho_{\text{cq}} = \sum_k p_k P_k^{\mathcal{A}} \otimes \rho_k^{\mathcal{B}}, \quad (\text{III.15})$$

where $\{P_k^{\mathcal{A}}\}$ is a von Neumann measurement on \mathcal{A} , $\{\rho_k^{\mathcal{B}}\}$ are arbitrary quantum states on \mathcal{B} and $\{p_k\}$ is the classical probability distribution. One can also consider fully classically correlated quantum states. Such "classical - classical" states ρ_{cc} can be written as

$$\rho_{\text{cc}} = \sum_{j,k} p_{jk} P_j^{\mathcal{A}} \otimes P_k^{\mathcal{B}}, \quad (\text{III.16})$$

where $\{P_j^{\mathcal{A}}\}$ and $\{P_k^{\mathcal{B}}\}$ are von Neumann measurements on \mathcal{A} and \mathcal{B} respectively and $\{p_{jk}\}$ is a two - dimensional probability distribution. In the case of two qubits, these states are defined in terms of orthogonal projectors

$$P_1 = \begin{pmatrix} \cos^2 \vartheta & \frac{1}{2}e^{-i\varphi} \sin 2\vartheta \\ \frac{1}{2}e^{i\varphi} \sin 2\vartheta & \sin^2 \vartheta \end{pmatrix}, \quad (\text{III.17})$$

$$P_2 = \begin{pmatrix} \sin^2 \vartheta & -\frac{1}{2}e^{-i\varphi} \sin 2\vartheta \\ -\frac{1}{2}e^{i\varphi} \sin 2\vartheta & \cos^2 \vartheta \end{pmatrix}.$$

In particular

$$\rho_{\text{cq}} = p_1 P_1 \otimes \rho_1^{\mathcal{B}} + p_2 P_2 \otimes \rho_2^{\mathcal{B}}, \quad (\text{III.18})$$

where

$$\rho_{1,2}^{\mathcal{B}} = \frac{1}{2}(\mathbb{1} + \mathbf{a}_{1,2} \cdot \boldsymbol{\sigma}), \quad (\text{III.19})$$

and the vectors $\mathbf{a}_{1,2} \in \mathbb{R}^3$ satisfy $\|\mathbf{a}_{1,2}\| \leq 1$. Similarly, up to the local unitary equivalence, the states ρ_{cc} can be written as

$$\rho_{cc} = \sum_{j,k=1}^2 p_{jk} P_j \otimes P_k. \quad (\text{III.20})$$

Since the states ρ_{cq} and ρ_{cc} have zero discord only maximal mutual correlation and correlation distance can be greater than zero.

Let us start with classical - quantum state. By a direct calculations one shows that

$$C_M(\rho_{cq}) = 2p_1p_2 \|\mathbf{a}_1 - \mathbf{a}_2\|, \quad (\text{III.21})$$

where $\|\cdot\|$ is the euclidean norm in \mathbb{R}^3 . The factor $2p_1p_2$ in (III.21) can be interpreted in terms of the statistical properties of a discrete classical random variable. Let X be the random variable with values in the set $\{-1, +1\}$ and probability distribution

$$\text{Prob}\{X = +1\} = p_1, \quad \text{Prob}\{X = -1\} = p_2. \quad (\text{III.22})$$

Then

$$(\text{Var}X)^2 = 4p_1p_2 \quad (\text{III.23})$$

and

$$C_M(\rho_{cq}) = \frac{1}{2} (\text{Var}X)^2 \|\mathbf{a}_1 - \mathbf{a}_2\|. \quad (\text{III.24})$$

Notice that in this case maximal mutual correlation depends on the classical probability distribution defining the state ρ_{cq} but also contains some information about the properties of quantum states of the subsystem \mathcal{B} . The largest possible value of $C_M(\rho_{cq})$ is achieved for $p_1 = p_2 = 1/2$ and when the Bloch vectors $\mathbf{a}_1, \mathbf{a}_2$ are normalized and satisfy $\mathbf{a}_2 = -\mathbf{a}_1$ i.e. when the states $\rho_1^{\mathcal{B}}, \rho_2^{\mathcal{B}}$ are projectors on orthogonal one - dimensional subspaces. But such state is the instance of classical - classical state (III.20), so for the genuine classical - quantum state $C_M(\rho_{cq}) < 1$.

In the case of general classical - classical state (III.20) the calculations show that $C_M(\rho_{cc})$ depends only on the properties of classical probability distribution $\{p_{jk}\}$. One can check that

$$C_M(\rho_{cc}) = |(p_{11} - p_{22})^2 - (p_{12} - p_{21})^2 + p_{12} + p_{21} - p_{11} - p_{22}|. \quad (\text{III.25})$$

In this case, the right hand side of (III.25) can be interpreted as the modulus of covariance of two discrete random variables X and Y with values in the set $\{-1, +1\}$. This can be proved by considering the following joint probability distribution of X and Y

$$\begin{aligned} \text{Prob}\{X = +1, Y = +1\} &= p_{11}, \\ \text{Prob}\{X = +1, Y = -1\} &= p_{12}, \\ \text{Prob}\{X = -1, Y = +1\} &= p_{21}, \\ \text{Prob}\{X = -1, Y = -1\} &= p_{11}. \end{aligned} \quad (\text{III.26})$$

Since

$$\begin{aligned} \langle X \rangle &= p_{11} + p_{12} - p_{21} - p_{22}, \\ \langle Y \rangle &= p_{11} + p_{21} - p_{12} - p_{22} \end{aligned} \quad (\text{III.27})$$

and

$$\langle XY \rangle = p_{11} + p_{22} - p_{12} - p_{21}, \quad (\text{III.28})$$

then

$$\begin{aligned} \text{Cov}(X, Y) &= p_{11} + p_{22} - p_{12} - p_{21} \\ &\quad + (p_{12} - p_{21})^2 - (p_{11} - p_{22})^2. \end{aligned} \quad (\text{III.29})$$

So

$$C_M(\rho_{cc}) = |\text{Cov}(X, Y)|. \quad (\text{III.30})$$

Concerning the correlation distance, the states ρ_{cq} and ρ_{cc} have interesting property: in both cases correlation distance equals to maximal mutual correlation

$$C_M(\rho_{cq}) = C(\rho_{cq}), \quad C_M(\rho_{cc}) = C(\rho_{cc}). \quad (\text{III.31})$$

In particular $C(\rho_{cc})$ is given by covariance of classical random variables.

C. Some separable states with non - zero discord

Consider now the following class of two qubit states

$$\rho_d = \begin{pmatrix} w & 0 & 0 & s \\ 0 & w & s & 0 \\ 0 & s & \frac{1}{2} - w & 0 \\ s & 0 & 0 & \frac{1}{2} - w \end{pmatrix}, \quad (\text{III.32})$$

where

$$0 < w < \frac{1}{2}, \quad 0 < s \leq s_{\max} \quad (\text{III.33})$$

and

$$s_{\max} = \sqrt{\frac{1}{2}w - w^2}. \quad (\text{III.34})$$

The states (III.32) are separable, but

$$D_1(\rho_d) = \frac{4s|1 - 4w|}{\sqrt{16s^2 + (1 - 4w)^2}}. \quad (\text{III.35})$$

Thus for all $w \in (0, 1/2)$ except of $w = 1/4$ and admissible s , ρ_d has non - zero discord. On the other hand, for such states the covariance matrix Q is very simple and equals to

$$Q = \begin{pmatrix} 4s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{III.36})$$

so

$$C_M(\rho_d) = C(\rho_d) = 4s \quad (\text{III.37})$$

and

$$D_1(\rho_d) < C_M(\rho_d) = C(\rho_d) \quad (\text{III.38})$$

D. Some entangled states

Now we take the states which are entangled. Consider the following family of two qubit states [13]

$$\rho_{\vartheta} = \begin{pmatrix} \frac{1}{2}\cos^2\vartheta & 0 & 0 & \frac{1}{4}\sin 2\vartheta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4}\sin 2\vartheta & 0 & 0 & \frac{1}{2}\sin^2\vartheta \end{pmatrix}, \quad (\text{III.39})$$

where $\vartheta \in (0, \pi/2)$. For this family we have

$$N(\rho_{\vartheta}) = \frac{\sqrt{6-2\cos 4\vartheta}-2}{4} \quad (\text{III.40})$$

and

$$D_1(\rho_{\vartheta}) = \frac{1}{2}\sin 2\vartheta. \quad (\text{III.41})$$

Moreover, the covariance matrix equals to

$$Q = \begin{pmatrix} \frac{1}{2}\sin 2\vartheta & 0 & 0 \\ 0 & -\frac{1}{2}\sin 2\vartheta & 0 \\ 0 & 0 & \frac{1}{4}\sin^2 2\vartheta \end{pmatrix}, \quad (\text{III.42})$$

so

$$C_M(\rho_{\vartheta}) = \frac{1}{2}\sin 2\vartheta \quad (\text{III.43})$$

and

$$C(\rho_{\vartheta}) = \frac{1}{2}\sin 2\vartheta + \frac{1}{8}\sin^2 2\vartheta. \quad (\text{III.44})$$

Thus

$$N(\rho_{\vartheta}) < D_1(\rho_{\vartheta}) = C_M(\rho_{\vartheta}) < C(\rho_{\vartheta}) \quad (\text{III.45})$$

for all $\vartheta \in (0, \pi/2)$.

E. General class of Bell - diagonal states

Consider now the states of the form

$$\rho_{BD} = \frac{1}{4} \left(\mathbb{1} \otimes \mathbb{1} + \sum_{j=1}^3 c_j \sigma_j \otimes \sigma_j \right), \quad (\text{III.46})$$

where $c = (c_1, c_2, c_3)$ is a three - dimensional real vector. Define also

$$c_+ = \max(|c_1|, |c_2|, |c_3|) \quad (\text{III.47})$$

and

$$c_0 = \min(|c_1|, |c_2|, |c_3|) \quad (\text{III.48})$$

where (III.48) represents intermediate among the numbers $|c_1|$, $|c_2|$ and $|c_3|$. It is known that [11]

$$D_1(\rho_{BD}) = c_0 \quad (\text{III.49})$$

and

$$N(\rho_{BD}) \leq D_1(\rho_{BD}). \quad (\text{III.50})$$

On the other hand, corresponding correlation matrix Q is also diagonal and reads

$$Q = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}, \quad (\text{III.51})$$

so

$$C_M(\rho_{BD}) = c_+. \quad (\text{III.52})$$

Since $c_0 \leq c_+$, we obtain

$$N(\rho_{BD}) \leq D_1(\rho_{BD}) \leq C_M(\rho_{BD}) \leq C(\rho_{BD}). \quad (\text{III.53})$$

IV. CONCLUSIONS

In this note we have studied bipartite correlations in the algebraic formulation of quantum mechanics. The proposed measure of correlations, called maximal mutual correlation, is a generalization of a natural measure of nonseparability of pure states. In the case of pure qubit states, such measure equals to entanglement, but for mixed states it can be non - zero also for classically correlated quantum states. As we have shown on explicit examples, maximal mutual correlation is greater or equal to quantum discord and is always dominated by the correlation distance.

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